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Lyapunov exponent analysis in an one dimensional map and BVP equation

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Abstract : We analyse the probability distribution of local expansion rate of chaotic attractor of an one dimensional map and a continuous dynamical system, the Bonhoeffer-van der Pol (BVP) equation We show that the distribution of largest local expansion rate is invariant in these systems We have studied the statistical dynamics of the local Lyapunov exponent $\lambda(X, L)$, calculated after every L steps. The standard deviation of $\lambda(X, L)$ about the mean $\bar{\lambda}$ (L) and Allan variance are found to approach zero in the limit $L \to \infty$ as $L^{-\alpha}$. We find that $\alpha \approx 0.5$ for the chaotic attractor of the one dimensional map and is ≈ 0.85 for the BVP chaotic attractor.

Keywords : One dimensional map, BVP equation, chaos and Lyapunov exponent

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1. Introduction

Chaotic motion is extremely sensitive to small variations in initial conditions. In a chaotic regime two nearby trajectories diverge exponentially until they become completely uncorrelated. The exponential divergence is characterized by the positive Lyapunov exponent. In fact, the spectrum of Lyapunov exponents [1] has proven to be one of the most useful diagonostics to detect and quantify regular and chaotic motions. For a chaotic motion at least one Lyapunov exponent must be positive. Recently, the study of statistical dynamics of local expansion rate on chaotic attractor has received considerable interest [2–7].

In this paper, we investigate certain features associated with the variations of local expansion rate $\lambda(X_t)$ and local exponents $\lambda(X, L)$ calculated after L time steps in a discrete system and a continuous dynamical system. The systems we study are the one dimensional map [6]

$$X_{n+1} = X_n \exp [A (1 - X_n)] \tag{1}$$

and the Bonhoeffer-van der Pol (BVP) equation [8]

$$\dot{x} = x - x^3/3 - y + f\cos t, \tag{2a}$$

$$\dot{y} = c(x + a - by), \tag{2b}$$

where a, b, c are constant parameters and f is the amplitude of the membrane current. Very recently, we have shown that singular local structures of chaotic attractors of (1) and (2) at various bifurcations can be characterized in terms of q-phase transitions of coarse-grained expansion rates of nearby orbits [6,7]. In the present paper we show that invariant spectrum of local expansion rate exist for chaotic attractors of (1) and (2). The standard deviation μ of $\lambda(X, L)$ about the mean Lyapunov exponent $\bar{\lambda}(L)$ and the Allan variance σ^2 are found to approach zero as $L \to \infty$. Both μ and σ goes to zero as $\sim L^{-\alpha}$ where α is a scaling exponent.

The paper is organised as follows. In Section 2, first we study the probability distribution of local expansion rate $P(\lambda)$ in the map (1). We show that $P(\lambda)$ is invariant. Then we present our analysis on BVP oscillator (2). In Section 3, we analyse the variation of local Lyapunov exponent about its mean value for both the systems (1) and (2). The scaling behaviour associated with standard deviation and Allan variance are investigated. Finally Section 4 contains conclusions.

2. Invariant spectrum of local expansion rate

For an orbit $\{X_i\}$, $(i = 1, 2, ..)$ of an one dimensional map $X_{i+1} = F(X_i)$ the one dimensional local expansion rate $\lambda(X_i)$ is given by

$$\lambda(X_i) = \ln |F'(X_i)|. \tag{3}$$

The average of mean Lyapunov exponent $\bar{\lambda}$ is then obtained from

$$\bar{\lambda} = (1/N) \sum_{i=1}^{N} \lambda(X_i). \tag{4}$$

For the map (1) eq. (3) takes the form

$$\lambda(X_i) = \ln |(1 - AX_i) \exp [A(1 - X_i)]|. \tag{5}$$

We fix A at 2.8 for which chaotic attractor is found [6]. The local expansion rate is calculated after every one iteration of the map. Figure 1a displays the probability distribution $P(\lambda)$ of λ calculated for the first 2×10^4 iterations while Figure 1b corresponds to next 2×10^4 iterations. Further, the local expansion rate is calculated for one iteration starting from randomly chosen 2×10^4 initial conditions on the chaotic attractor and the corresponding $P(\lambda)$ is represented by dots in Figure 1b. We immediately see that $P(\lambda)$ are almost identical in all the three cases. That is, $P(\lambda)$ is invariant for a particular orbit and also for all orbits in the same chaotic region. $P(\lambda)$ is independent of the value of X_1. The above analysis is also performed for the BVP chaotic attractor. For the BVP equation the three local expansion rates are calculated employing the algorithm described by Wolf *et al* [9].

Eq. (2) together with the variational equations are integrated with time step 0.01 and the local expansion rates are calculated after every one time step. In our analysis we consider

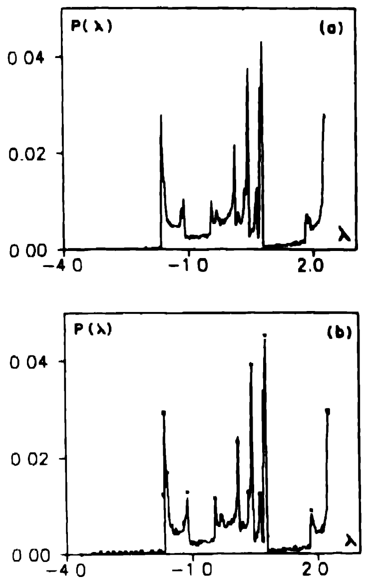

Figure 1. (a) The probability distribution $P(\lambda)$ of $\lambda(X_i)$ for the chaotic attractor of the map (1) with $A = 2.8$. This is for first 2×10^4 iterations. (b) The full curve gives $P(\lambda)$ resulting from the next 2×10^4 iterations. The dots give the $P(\lambda)$ for randomly chosen 2×10^4 initial conditions. The three distributions practically coincide.

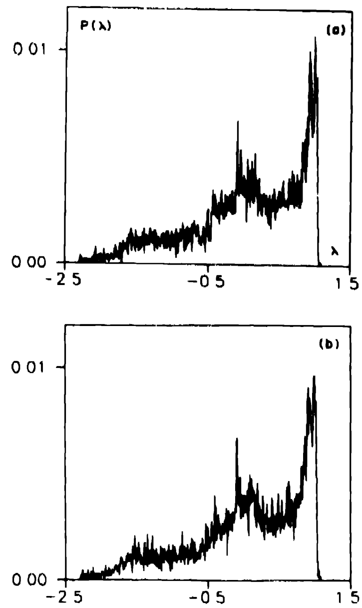

Figure 2. The probability distribution $P(\lambda)$ of $\lambda(X(t))$ for the chaotic attractor of the BVP eq. (2). (a) gives $P(\lambda)$ from the first 2×10^4 data and (b) from the next 2×10^4 data.

only the maximal expansion rate $\lambda(X(t))$ where $X = (x, y)$. In the BVP eq. (2) chaotic attractor is found for $a = 0.7$, $b = 0.8$, $c = 0.1$ and $f = 0.74$ [8]. The corresponding invariant spectrum is shown in Figure 2. Figure 2a is obtained from the first 2×10^4 data and Figure 2b results from the next 2×10^4 data. The two figures are almost identical. From the Figures 1 and 2 we see that $\lambda(X)$ vary significantly over the chaotic attractors. Further, it is clear that $P(\lambda)$ is invariant and is system dependent.

Usually $\bar{\lambda}$ is calculated by following an orbit for an extremely long time. However, Figure 1b shows that alternatively one can obtain accurate value of Lyapunov exponent for a chaotic attractor by calculating local expansion rate, for a short time, for many orbits starting from various initial conditions chosen on the attractor and taking average value of them. This kind of analysis can be used for systems which evolves on time scales of days and weeks and for systems which cannot be practically followed for a long time. For example, the chaotic evolution of stress variable in a plastic instability experiment [10,11] terminate in a finite time and hence a sufficiently long term analysis is practically impossible. In such a situation local expansion rate calculated by performing the experiment for a short time starting from various initial conditions can be used to obtain Lyapunov exponent.

3. Variation of local Lyapunov exponent

We consider the local Lyapunov exponent $\lambda(X_i, L)$ calculated after L time steps from X_i. For the map (1) it is defined as

$$\lambda(X_i, L) = (1/L) \sum_{j=0}^{L-1} \ln\left|F'(X_{j+i})\right|. \tag{6}$$

Then $\bar{\lambda}(L)$ is given by

$$\bar{\lambda}(L) = (1/N) \sum_{i=1}^{N} \lambda(X_i, L), \tag{7}$$

where N is large. The local Lyapunov exponent $\lambda(X_i, L)$ is relevant when one wants to know the predictability of a system either locally in the phase space of the attractor or by the variation of the average exponent over the attractor.

We study the variation of the local exponent $\lambda(X, L)$ by the probability distribution function $P(\lambda, L)$ The $P(\lambda, L)$ gives the normalized number of times any value of one of $\lambda(X, L)$ appears as we vary X on the attractor for a fixed L. Figure 3 shows the distribution

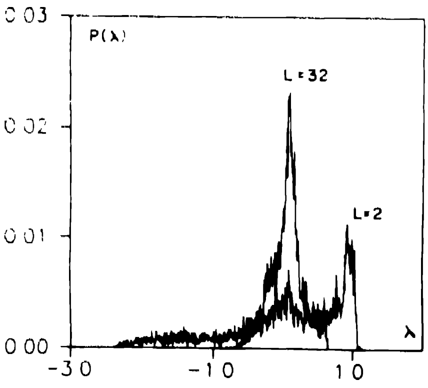

Figure 3. The probability distribution of $\lambda(X, L)$ for the map (1) for $L = 2$ and 32.

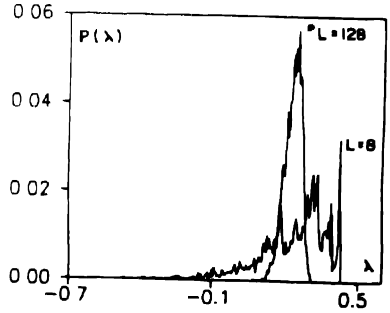

Figure 4. $P(\lambda, L)$ for the BVP eq. (2)

curve $P(\lambda, L)$ for $L = 2$ and 32 for the map (1). 10^4 data are used in our analysis. The distributions differ markedly. For small L the local Lyapunov exponent has a very broad distribution. Thus the growth of perturbations in time step $L = 2$ will be enormously different from point to point on the attractor. As L becomes larger its distributions over the attractor sharpens and peaks around the mean value $\bar{\lambda}(L)$ as is the case for $L = 32$. The $P(\lambda, L)$ of the BVP equation is given in the Figure 4. Here again extremely wide distribution of $\lambda(X, L)$ is found for small L while small variations around $\bar{\lambda}$ is found for large L.

From $P(\lambda, L)$ one can extract important information by examining its mean value and the variance about the mean Lyapunov exponent $\bar{\lambda}$. The moments about $\bar{\lambda}(L)$ are given by

$$\mu^2(p,L) = (1/N)\sum_{i=1}^{N}\left[\lambda(X_i,L) - \bar{\lambda}(L)\right]^p. \tag{8}$$

$\mu^2(2, L)$ is the variance about the mean. Its square root is the standard deviation. Allan variance $\sigma^2(2, L)$ is also of useful and is defined as

$$\sigma^2(2,L) = (1/2N)\sum_{i=1}^{N}\left[\lambda(X_{i+1},L) - \lambda(X_i,L)\right]^2. \tag{9}$$

We have limited ourselves to the following values of L : 2^i, $i = 1, 2, \ldots, 9, 10$. Figure 5 shows the numerically calculated $\mu(2, L)$ and $\sigma(2, L)$ as a function of L for the map (1) with

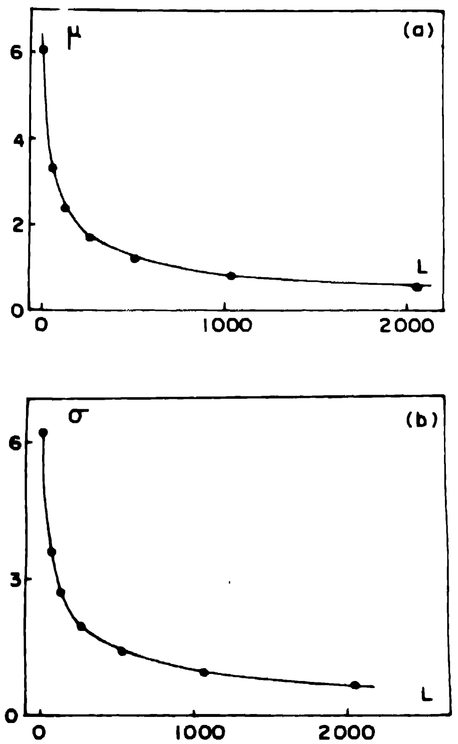

Figure 5. (a) $\mu(2, L)$ *versus* L for the map (1) with $A = 2.8$. The dots mark numerically calculated points and the solid line is a best fit with a power law dependence. (b) $\sigma(2, L)$ *versus* L.

$A = 2.8$. $\mu(2, L)$ and $\sigma(2, L)$ approach to zero as $L \to \infty$. In Figure 6, we plotted $\mu(2, L)$ *versus* L and $\sigma(2, L)$ *versus* L in the $\log_2 - \log_2$ scale. The dot marks calculated points and the solid line is the best straight line fit. From the Figure 6, we found $\mu(2, L) = 23.71$

$L^{-0.46}$. We have studied fourth root of $\mu^2(4, L)$ and $\sigma^2(4, L)$. They have also behaved as an inverse powers of L. We found that $\sqrt{\mu(4, L)} = 3.95L^{-0.48}$ and $\sqrt{\sigma(4, L)} = 5.72L^{-0.52}$.

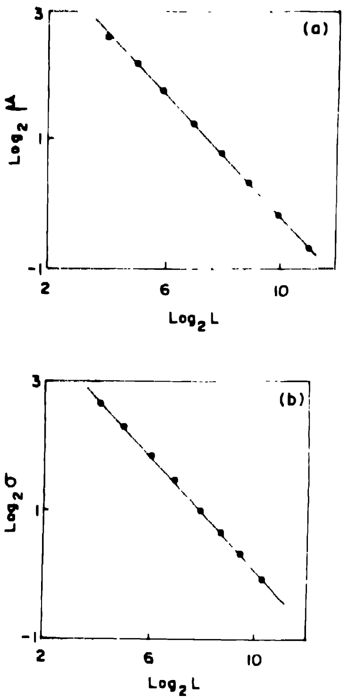

Figure 6. Same as Figure 5 except in $\log_2$ - $\log_2$ scale

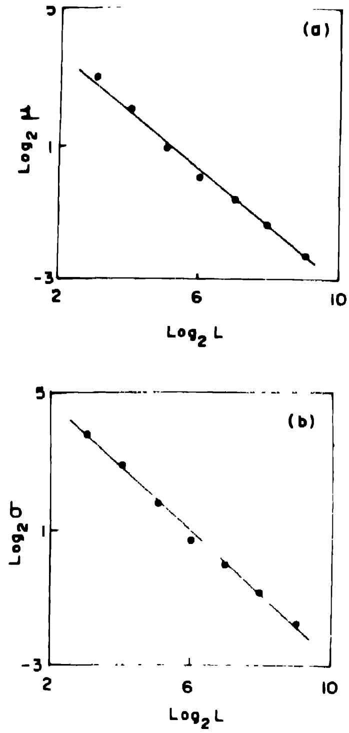

Figure 7. $\mu(2, L)$ *versus* $L(a)$ and $\sigma(2, L)$ *versus* $L(b)$ in $\log_2$ – $\log_2$ scale for the BVP eq (2) with $a = 0.7$, $b = 0.8$, $c = 0.1$ and $f = 0.74$

For the BVP equation we have calculated the largest local Lyapunov exponent for a set of values of L. Figure 7 displays the numerically calculated $\mu(2, L)$ and $\sigma(2, L)$ for the BVP chaotic attractor. Here again, we find that $\mu(2, L)$ and $\sigma(2, L)$ shrink to zero as inverse powers of L, here $\mu(2, L) = 39.11L^{-0.84}$ and $\sigma(2, L) = 7.6L^{-0.88}$.

4. Conclusions

In this paper we have focussed on the variation of the maximal Lyapunov exponent in the map (1) and the BVP eq. (2). The distribution of local expansion rate in these systems is found to invariant. Further, the variations of local expansion rate about the mean Lyapunov exponent approach zero as the number of steps $L \to \infty$. This approach is as $L^{-\alpha}$ and α is much higher for the BVP eq. (2) than the map (1).

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